Mathematical Modelling Presentation

What Goes Around Comes Around

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1 Introduction

Around the early 1600's a brilliant man by name of Johannes Kepler observed the nights sky and drew some very significant conclusions. With the aid of detailed observations from his mentor, Tycho Brahe, Kepler came up with three laws concerning planetary motion.

1) The orbit of a planet is an ellipse with the sun (centre of attraction) at the focus

2) The radius vector from the sun to a planet sweeps out equal areas in equal intervals of time

3) The orbital period squared is proportional to the semi-major axis cubed

$$T^2 \propto a^3$$

Not more than a century later, Newton took it upon himself to build a mathematical framework for Kepler's laws. In his famous publication: 'Principia Mathematica', Newton lays out his famous theory of motion. Newton's theory was developed with the ideas of force and motion in mind.

The direct problem would be to determine the motion of an object (effect) from the forces acting upon it (cause) and the inverse problem would be to determine the force given the motion.

Kepler's laws are empirical, meaning that they were only based in observation and that there was not yet an underlying theory to describe the behaviour of planetary bodies. It became necessary for Newton to solve the inverse problem, which would include determining the forces causing the motion of planetary bodies. Once he found this out, he could establish generalized laws that could be applied to solve direct problems in astronomy, classical mechanics etc.

Our specific objective will be to solve the inverse problem of Kepler's first and second law, using his third as a kind of stepping stone as it it is already expressed in the form of a convenient equation.

2 Jumping off with Kepler's Third Law

Newton started off with contemplating how the moon was able to stay in orbit under the influence of gravity. He assumed the moon's orbit was circular and that the speed was

$$v = \frac{2\pi r}{T} \tag{1}$$

where $2\pi r$ is the total orbit length and T is the period of revolution, which will give us units of meters per second. Newton then reasoned that the force holding the moon in orbit was the centripetal force

$$F_{moon} = \frac{v^2}{r} \tag{2}$$

which you can work through at home by converting to polar and doing a couple derivatives on the position vector.

Assuming Kepler's third law holds for the moon (in circular orbit) as it does for other planets, this means that

$$T^2 \propto \underbrace{r^3}_{circle} \tag{3}$$

Now we can follow the line of reasoning below

$$v^2 \propto \frac{r^2}{T^2} \propto \frac{r^2}{r^3} \propto r^{-1} \tag{4}$$

and dividing through by r will give us

$$F_{moon} \propto r^{-2} \tag{5}$$

which is the famous inverse square law which suggests that the force holding planetary bodies in place is based on the distance at which the body is away from the centre of attraction. Closer to the centre, more attractive force, further away, less attractive force.

3 Kepler's Second Law

After the discovery of the inverse square law, Newton theorized the that the force attracting a planet to the sun would be a central force which is only a function of distance, r.

$$\ddot{\hat{r}} = -f(r)\frac{\hat{r}}{r} \tag{6}$$

with the negative indicating that force is directed towards the centre.

From this assumption we have two consequences:

1) The cross product of the position vector with its second derivative (centripetal acceleration) is zero because they lie on top of each other

$$\hat{r} \times \ddot{\hat{r}} = -f(r)\hat{r} \times \hat{r} = 0 \tag{7}$$

2) The time derivative of the position vector and its derivative is also zero using the result above

$$\frac{d}{dt}(\hat{r} \times \dot{\hat{r}}) = \underbrace{\dot{\hat{r}} \times \dot{\hat{r}}}_{\text{0, same vector}} + \underbrace{\dot{r} \times \ddot{\hat{r}}}_{\text{0, from above}} = 0 \tag{8}$$

The cross product between the position vector and its derivative is known as angular momentum. Since the value of angular momentum does not change with time, angular momentum is said to be conserved and is represented with a constant c

$$\hat{r} \times \dot{\hat{r}} = \hat{c} \tag{9}$$

This means that motion of the planetary body is constrained to a plane that has no torque and is perpendicular to the vector c.

It's convenient for us to work with polar coordinates in a planar system so we can write the position vector as

$$\hat{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j} \tag{10}$$

with its derivative

$$\dot{\hat{r}} = (\dot{r}\cos\theta - r\sin\theta\dot{\theta})\hat{i} + (\dot{r}\sin\theta + r\cos\theta\dot{\theta})\hat{j} \tag{11}$$

where theta is the angle from $\hat{i}$ to $\hat{r}$.

The area swept out by a radius vector from 0 to theta is

$$A = \frac{1}{2}\int_0^\theta r(\alpha)^2 d\alpha \tag{12}$$

meaning that the rate at which area is swept out, by the fundamental theorem of calculus, is

$$\frac{dA}{dt} = \frac{1}{2}r(\theta)^2\dot{\theta} \tag{13}$$

If we compute the conservation of angular momentum using the the new polar position vector and its derivative, we will arrive at

$$\hat{c} = \hat{r} \times \dot{\hat{r}} = 2\frac{dA}{dt}\underbrace{\frac{\hat{c}}{c}}_{\hat{k}} \tag{14}$$

meaning change in area swept out over with time at any interval is

$$\frac{dA}{dt} = \frac{c}{2} \tag{15}$$

A CONSTANT!!! And thus we have arrived at Kepler's Second Law.

4 Kepler's First Law

Before we begin with Kepler's first law, let us first define what a conic section is.

A conic section is a plane curve, meaning that it is the line intersection of a plane through a cone. The type of conic section varies based on the orientation of the plane in the cone.

Conic sections are curves in which the ratio of the distance from a fixed point to a point on the curve (OP) to the distance from the point on the curve to the directrix (PQ) is a constant which we call eccentricity.

That was a mouthful, but the equation and the diagram sum it up nicely:

$$\frac{|OP|}{|PQ|} = e \tag{16}$$

Rearranging and subbing in a few variables from the diagram, we get

$$r = \frac{ek}{1 + e\cos\theta} \tag{17}$$

which is the general polar equation used to represent conic sections.
Now that we know what a conic section is, let's see if Newton's central force theory holds up.
This time we will apply the inverse-square function to the central force equation which will leave us with

$$\ddot{\hat{r}} = -\frac{a}{r^3}\hat{r} \tag{18}$$

where a is the semi-major axis and is therefore a constant.
Once again, we will bring out conservation of angular momentum because we know it holds under a central force

$$\hat{r} \times \dot{\hat{r}} = \hat{c} \tag{19}$$

if we take the cross product of c with the position vector and divide by r^3 we get the following equation:

$$\frac{d}{dt}\left(\frac{\hat{r}}{r}\right) = \frac{\hat{c} \times \hat{r}}{r^3} \tag{20}$$

which you can verify at home using the modified identity $a \times (b \times a) = (a \cdot a)b - (b \cdot a)a$
Multiplying through by a factor of $-a$ and integrating with respect to time, we arrive at

$$a(\hat{r} \cdot \hat{e} + r) = c^2 \tag{21}$$

Note that $\hat{e}$ appears and this is because it was the constant of integration in the previous step. This is known as the eccentricity vector and it lies in the orbital plane. If we measure angular displacement of the orbiting body with respect to the eccentricity vector, it will give

$$\hat{r} \cdot \hat{e} = re\cos\phi \tag{22}$$

subbing in and rearranging we arrive at

$$r = \frac{ek}{1 + e\cos\phi} \tag{23}$$

where $k = \dfrac{c^2}{(ae)}$

which is the polar equation for a conic section. Therefore, not only do planetary bodies orbit in ellipses, they can orbit in the path of any of the conic sections depending on the value of e. Here we have Kepler's first law.

5 Impact and benefits

Now that we have solved the inverse problem of using Kepler's observations to build the theory of motion, we can now go about solving direct problems!

Of course, the most obvious application would be orbital mechanics. Kepler's First Law states that planetary bodies orbit in ellipses, but we now know we can extend and say that an inverse-square central force will result in conic section trajectory.

For example: the moon is a nearly circular orbit and has an eccentricity of 0.054 while something like Halley's comet (which is in orbit) has an high eccentricity of 0.967 which corresponds to an orbit that is highly elliptical.

Circle, e = 0
Ellipse, 0 ¡ e ¡ 1
Parabola, e = 1
Hyperbola, e ¿ 1

Knowing the polar equation for conic sections allows us to predict the motion of bodies in parabolic and hyperbolic trajectories as well depending on the value of eccentricity.

The diagram shows the four different types of conic sections with V_c representing circular orbital speed and V_e representing escape speed. You can see that the circular and elliptical orbits are bound while the parabolic trajectory is $V = V_e$ and the hyperbolic trajectory is $V > V_e$. The parabolic trajectory corresponds to something a comet that has no initial velocity while the hyperbolic has a starting energy that will curve its trajectory a certain way.

A specific application would be to a rocket's trajectory in a gravity assist, which is a procedure that utilizes gravity from an astronomical object to alter its path. An extreme example is how a rocket trajectory can be altered to go in the opposite direction. This can be done by accelerating at the correct time and giving the ship an initial velocity such that the rocket will follow a hyperbolic trajectory around, say, a moon.

The inverse-square central force can also be seen in other areas of science including electrostatics in Coulomb's law as well as light intensity in waves and optics.

Thank you for your time.